

Plasma Wave Properties of the Schwarzschild Magnetosphere in a Veselago Medium

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Abstract

We re-formulate the 3+1 GRMHD equations for the Schwarzschild black hole in a Veselago medium. Linear perturbation in rotating (non-magnetized and magnetized) plasma is introduced and their Fourier analysis is considered. We discuss wave properties with the help of wave vector, refractive index and change in refractive index in the form of graphs. It is concluded that some waves move away from the event horizon in this unusual medium. We conclude that for the rotating non-magnetized plasma, our results confirm the presence of Veselago medium while the rotating magnetized plasma does not provide any evidence for this medium.

Keywords: Veselago medium; 3+1 formalism; GRMHD equations; Isothermal plasma; Dispersion relations.

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1 Introduction

Schwarzschild black hole is the simplest one which has neither angular momentum nor charge. Plasma is the most common form of matter in the

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universe. It can be accelerated and directed by electromagnetic fields which make it to be controlled and employed. The matter in inner-stellar space is composed of plasma. The general relativistic magnetohydrodynamics (GRMHD) is the standard theory which is helpful to explore the dynamics of falling magnetized plasma towards the event horizon.

An accretion disk is formed near the Schwarzschild black hole as plasma flows towards it. Perturbations in black hole regime have always been fascinated the researchers. Regge and Wheeler [1] discussed the stability of Schwarzschild singularity. Price [2] examined the dynamics of non-spherical perturbations during the collapse of stars with a scalar-field analog. Waves in an electron-positron plasma were investigated by Sakai and Kawata [3] in a frame of two-fluid equation for the Schwarzschild black hole. Southwood [4] used a small perturbation approach to derive the general stability criteria for plasma with isotropic pressure. Gleiser et al. [5] studied even parity perturbation of the Schwarzschild black hole up to second order.

Arnowitt, Deser and Misner (ADM) [6] developed 3+1 formalism in which spacetime is foliated into layers of three-dimensional spacelike hypersurface, threaded by a timelike normal. Many relativists [7]-[8] applied this technique to illustrate different attributes of general relativity. Thorne and Macdonald [9]-[10] evolved the electromagnetic fields of the black hole theory. Holcomb and Tajima [11], Holcomb [12] and Dettmann et al. [13] analyzed the production of waves in the Friedmann universe. Buzzi et al. [14] examined the wave properties of two fluid plasma in the surroundings of Schwarzschild event horizon. Zhang [15] explicated the theory of stationary symmetric GRMHD in black hole regime. The same author [16] also studied the behavior of perturbation of cold plasma in the magnetosphere of Kerr black hole. Sharif and his collaborators [17]-[20] determined the plasma wave properties by using dispersion relations. They analyzed it for cold, isothermal and hot plasmas.

Veselago medium or Double negative medium (DNG) is the most famous class of metamaterials. This medium has simultaneously negative electric permittivity and magnetic permeability. Negative phase velocity (NPV) and negative refractive index are main features of this medium. Ziolkowski [21] analyzed electromagnetic wave properties in DNG. Valanju et al. [22] established the well-known idea that wave refraction in DNG medium is always positive but not homogenous. Ross et al. [23] found that NPV propagation develops at lower values of cosmological constant Λ in the vicinity of non-rotating black hole. Mackey and Lakhtakia [24] verified that presence

of charge raises the tendency of a rotating black hole to support the NPV propagation in its ergosphere. In a paper given in a book [25], J li generalized the concept of negative medium to acoustic waves. In a recent paper, Veselago [26] has explained how an electromagnetic wave transfers energy, linear momentum and mass in a negative refraction-medium.

In the present paper, we discuss isothermal plasma wave properties of the Schwarzschild black hole in a Veselago medium. The format of the paper is as follows: General line element and its formation in the Schwarzschild planar analogue is given in Section 2. In Section 3, perturbed and Fourier analyzed forms of 3+1 GRMHD equations for isothermal plasma are presented. Sections 4 and 5 contain the restricted 3 + 1 GRMHD equations for rotating (non-magnetized and magnetized) plasmas and discussion related to wave properties. Summary of the results is given in Section 6.

2 ADM 3+1 Formalism and Schwarzschild Planar Analogue

The general line element in *ADM 3+1 formalism* is given by [16]

$$ds^2 = -\alpha^2 dt^2 + \gamma_{ij}(dx^i + \beta^i dt)(dx^j + \beta^j dt), \quad (2.1)$$

where the lapse function α is the ratio of FIDO (fiducial observer) proper time to universal time, i.e., $\alpha = \frac{d\tau}{dt}$, β^i is the shift vector and γ_{ij} ($i, j = 1, 2, 3$) are the components of three dimensional absolute space. FIDO is the natural observer associated with the above line element. In Schwarzschild planar analogue, shift vector vanishes due to zero angular momentum and $\gamma_{ii} = 1$ represents Euclidean 3 geometry. Thus it reduces to the form [17]

$$ds^2 = -\alpha^2(z)dt^2 + dx^2 + dy^2 + dz^2, \quad (2.2)$$

where the directions z , x and y are analogous to the Schwarzschild coordinates r , ϕ and θ respectively.

3 3+1 GRMHD Equations for Isothermal Plasma in Veselago Medium

The 3 + 1 GRMHD equations in a Veselago medium for the plasma present in general line element and the Schwarzschild planar analogue (Eqs.(2.1) and

(2.2)) are given in Appendix A. In the vicinity of the Schwarzschild black hole, equation of state for isothermal plasma is [16]

$$\mu = \frac{\rho + p}{\rho_0}, \quad (3.1)$$

here ρ_0 , ρ , p and μ are the rest mass density, moving mass density, pressure and specific enthalpy respectively. The specific enthalpy μ is constant here. This equation indicates that there is no exchange of energy between the plasma and the magnetic field of the fluid. For isothermal plasma existing in Schwarzschild magnetosphere, the 3+1 GRMHD equations (A10)-(A14) take the form

$$\frac{\partial \mathbf{B}}{\partial t} = -\nabla \times (\alpha \mathbf{V} \times \mathbf{B}), \quad (3.2)$$

$$\nabla \cdot \mathbf{B} = 0, \quad (3.3)$$

$$\begin{aligned} & \frac{\partial(\rho + p)}{\partial t} + (\alpha \mathbf{V} \cdot \nabla)(\rho + p) + (\rho + p)\gamma^2 \mathbf{V} \cdot \frac{\partial \mathbf{V}}{\partial t} + (\rho + p)\gamma^2 \\ & V \cdot (\alpha \mathbf{V} \cdot \nabla) \mathbf{V} + (\rho + p) \nabla \cdot (\alpha \mathbf{V}) = 0, \end{aligned} \quad (3.4)$$

$$\begin{aligned} & \left\{ \left((\rho + p)\gamma^2 + \frac{\mathbf{B}^2}{4\pi} \right) \delta_{ij} + (\rho + p)\gamma^4 V_i V_j - \frac{1}{4\pi} B_i B_j \right\} \left(\frac{1}{\alpha} \frac{\partial}{\partial t} \right. \\ & \left. + \mathbf{V} \cdot \nabla \right) V^j - \left(\frac{\mathbf{B}^2}{4\pi} \delta_{ij} - \frac{1}{4\pi} B_i B_j \right) V_{,k}^j V^k + (\rho + p)\gamma^2 a_i + p_{,i} \\ & = \frac{1}{4\pi} (\mathbf{V} \times \mathbf{B})_i \nabla \cdot (\mathbf{V} \times \mathbf{B}) - \frac{1}{8\pi\alpha^2} (\alpha \mathbf{B})_{,i}^2 + \frac{1}{4\pi\alpha} (\alpha B_i)_{,j} B^j \\ & - \frac{1}{4\pi\alpha} [\mathbf{B} \times \{ \mathbf{V} \times (\nabla \times (\alpha \mathbf{V} \times \mathbf{B})) \}]_i, \end{aligned} \quad (3.5)$$

$$\begin{aligned} & \left(\frac{1}{\alpha} \frac{\partial}{\partial t} + \mathbf{V} \cdot \nabla \right) (\rho + p)\gamma^2 - \frac{1}{\alpha} \frac{\partial p}{\partial t} + 2(\rho + p)\gamma^2 (\mathbf{V} \cdot \mathbf{a}) + (\rho + p) \\ & \gamma^2 (\nabla \cdot \mathbf{V}) - \frac{1}{4\pi\alpha} (\mathbf{V} \times \mathbf{B}) \cdot (\mathbf{V} \times \frac{\partial \mathbf{B}}{\partial t}) - \frac{1}{4\pi\alpha} (\mathbf{V} \times \mathbf{B}) \cdot (\mathbf{B} \times \frac{\partial \mathbf{B}}{\partial t}) \\ & + \frac{1}{4\pi\alpha} (\nabla \times \alpha \mathbf{B}) = 0. \end{aligned} \quad (3.6)$$

We assume that plasma flows in xz -plane so that the velocity \mathbf{V} and magnetic field \mathbf{B} experienced by FIDO are given as

$$\begin{aligned} \mathbf{V} &= V(z)\mathbf{e}_x + u(z)\mathbf{e}_z, \\ \mathbf{B} &= B[\lambda(z)\mathbf{e}_x + \mathbf{e}_z], \end{aligned} \quad (3.7)$$

where B is an arbitrary constant. The relation among λ , u and V is given by [17]

$$V = \frac{V^F}{\alpha} + \lambda u, \quad (3.8)$$

where V^F is a constant of integration. $\gamma = \frac{1}{\sqrt{1-\mathbf{V}^2}}$ is the Lorentz factor which becomes

$$\gamma = \frac{1}{\sqrt{1-u^2-V^2}}. \quad (3.9)$$

The gravity of black hole perturbs the plasma flow. Inserting the linear perturbation to density ρ , pressure p , velocity \mathbf{V} and magnetic field \mathbf{B} , we have

$$\begin{aligned} \rho &= \rho^0 + \delta\rho = \rho^0 + \rho\tilde{\rho}, & p &= p^0 + \delta p = p^0 + p\tilde{p}, \\ \mathbf{V} &= \mathbf{V}^0 + \delta\mathbf{V} = \mathbf{V}^0 + \mathbf{v}, & \mathbf{B} &= \mathbf{B}^0 + \delta\mathbf{B} = \mathbf{B}^0 + B\mathbf{b}, \end{aligned} \quad (3.10)$$

where unperturbed and linearly perturbed quantities are denoted by ρ^0 , p , \mathbf{V}^0 , \mathbf{B}^0 and $\delta\rho$, δp , $\delta\mathbf{V}$, $\delta\mathbf{B}$ respectively. The quantities $\tilde{\rho}$, \tilde{p} , v_x , v_z , b_x and b_z are dimensionless which we shall introduce for the perturbed quantities

$$\begin{aligned} \tilde{\rho} &= \tilde{\rho}(t, z), & \tilde{p} &= \tilde{p}(t, z), \\ \mathbf{v} &= \delta\mathbf{V} = v_x(t, z)\mathbf{e}_x + v_z(t, z)\mathbf{e}_z, \\ \mathbf{b} &= \frac{\delta\mathbf{B}}{B} = b_x(t, z)\mathbf{e}_x + b_z(t, z)\mathbf{e}_z, \end{aligned} \quad (3.11)$$

The perfect GRMHD equations (Eqs.(3.2)-(3.6)), after the insertion of linear perturbation from Eq.(3.11), become

$$\frac{\partial(\delta\mathbf{B})}{\partial t} = -\nabla \times (\alpha\mathbf{v} \times \mathbf{B}) - \nabla \times (\alpha\mathbf{V} \times \delta\mathbf{B}), \quad (3.12)$$

$$\nabla \cdot (\delta\mathbf{B}) = 0, \quad (3.13)$$

$$\begin{aligned} &\frac{\partial(\delta\rho + \delta p)}{\partial t} + (\alpha\mathbf{V} \cdot \nabla)(\delta\rho + \delta p) + (\rho + p)\gamma^2\mathbf{V} \cdot \frac{\partial\mathbf{v}}{\partial t} - \alpha(\rho + p) \\ &(\mathbf{v} \cdot \nabla \ln u) + \alpha(\rho + p)(\nabla \cdot \mathbf{v}) + (\delta\rho + \delta p)(\nabla \cdot \alpha\mathbf{V}) + (\delta\rho + \delta p)\gamma^2 \\ &\mathbf{V} \cdot (\alpha\mathbf{V} \cdot \nabla)\mathbf{V} + 2(\rho + p)\gamma^2(\mathbf{V} \cdot \mathbf{v})(\alpha\mathbf{V} \cdot \nabla) \ln \gamma + (\rho + p)\gamma^2 \\ &(\alpha\mathbf{V} \cdot \nabla\mathbf{V}) \cdot \mathbf{v} + (\rho + p)\gamma^2\mathbf{V} \cdot (\alpha\mathbf{V} \cdot \nabla)\mathbf{v} = 0, \end{aligned} \quad (3.14)$$

$$\begin{aligned}
& \left\{ \left((\rho + p)\gamma^2 + \frac{\mathbf{B}^2}{4\pi} \right) \delta_{ij} + (\rho + p)\gamma^4 V_i V_j - \frac{1}{4\pi} B_i B_j \right\} \frac{1}{\alpha} \frac{\partial v^j}{\partial t} \\
& + \frac{1}{4\pi} [\mathbf{B} \times \{ \mathbf{V} \times \frac{1}{\alpha} \frac{\partial(\delta\mathbf{B})}{\partial t} \}]_i + (\rho + p)\gamma^2 v_{i,j} V^j + (\rho + p) \\
& \times \gamma^4 V_i v_{j,k} V^j V^k - \frac{1}{4\pi\alpha} \{ (\alpha\delta B_i)_{,j} - (\alpha\delta B_j)_{,i} \} B^j + (\delta p)_i \\
& = -\gamma^2 \{ (\delta\rho + \delta p) + 2(\rho + p)\gamma^2 (\mathbf{V} \cdot \mathbf{v}) \} a_i + \frac{1}{4\pi\alpha} \{ (\alpha B_i)_{,j} \\
& - (\alpha B_j)_{,i} \} \delta B^j - (\rho + p)\gamma^4 (v_i V^j + v^j V_i) V_{k,j} V^k - \gamma^2 \{ (\delta\rho + \delta p) \\
& V^j + 2(\rho + p)\gamma^2 (\mathbf{V} \cdot \mathbf{v}) V^j + (\rho + p)v^j \} V_{i,j} - \gamma^4 V_i \{ (\delta\rho + \delta p) V^j \\
& + 4(\rho + p)\gamma^2 (\mathbf{V} \cdot \mathbf{v}) V^j + (\rho + p)v^j \} V_{j,k} V^k, \tag{3.15} \\
& \gamma^2 \frac{1}{\alpha} \frac{\partial(\delta\rho + \delta p)}{\partial t} + \mathbf{v} \cdot \nabla (\rho + p)\gamma^2 - \frac{1}{\alpha} \frac{\partial(\delta p)}{\partial t} + (\mathbf{V} \cdot \nabla) (\delta\rho + \delta p)\gamma^2 \\
& + 2(\rho + p)\gamma^4 (\mathbf{V} \cdot \nabla) (\mathbf{V} \cdot \mathbf{v}) + 2(\rho + p)\gamma^2 (\mathbf{v} \cdot \mathbf{a}) + 4(\rho + p)\gamma^4 (\mathbf{V} \cdot \mathbf{v}) \\
& (\mathbf{V} \cdot \mathbf{a}) + 2(\delta\rho + \delta p)\gamma^2 (\mathbf{V} \cdot \mathbf{a}) + (\rho + p)\gamma^2 (\nabla \cdot \mathbf{v}) + 2(\rho + p)\gamma^4 \\
& (\mathbf{V} \cdot \mathbf{v}) (\nabla \cdot \mathbf{V}) + (\delta\rho + \delta p)\gamma^2 (\nabla \cdot \mathbf{V}) = \frac{1}{4\pi\alpha} [\mathbf{v} \cdot (\mathbf{B} \cdot \frac{\partial \mathbf{B}}{\partial t}) \mathbf{V} + \mathbf{V} \cdot (\mathbf{B} \cdot \frac{\partial \mathbf{B}}{\partial t}) \mathbf{v} \\
& + \mathbf{V} \cdot (\mathbf{B} \cdot \delta \mathbf{B}) \mathbf{V} + \mathbf{V} \cdot (\delta \mathbf{B} \frac{\partial \mathbf{B}}{\partial t}) \mathbf{V} - \mathbf{v} \cdot (\mathbf{B} \cdot \mathbf{V}) \frac{\partial \mathbf{B}}{\partial t} - \mathbf{V} \cdot (\mathbf{B} \cdot \mathbf{V}) \frac{\partial \delta \mathbf{B}}{\partial t} \\
& - \mathbf{V} \cdot (\mathbf{B} \cdot \mathbf{v}) \frac{\partial \delta \mathbf{B}}{\partial t} - \mathbf{V} \cdot (\delta \mathbf{B} \cdot \mathbf{V}) \frac{\partial \mathbf{B}}{\partial t}] - \frac{1}{4\pi\alpha} [\mathbf{V} \cdot (\mathbf{B} \cdot \mathbf{B}) \frac{\partial \mathbf{v}}{\partial t} - \mathbf{V} \cdot (\mathbf{B} \cdot \frac{\partial \delta \mathbf{v}}{\partial t}) \mathbf{B}] \\
& - \frac{1}{4\pi\alpha} \nabla \times (\alpha \delta \mathbf{B}). \tag{3.16}
\end{aligned}$$

The component form of these equations, using Eq.(3.11), can be composed as follows

$$\begin{aligned}
& \frac{1}{\alpha} \frac{\partial b_x}{\partial t} - u b_{x,z} = (u b_x - V b_z - v_x + \lambda v_z) \nabla \ln \alpha \\
& - (v_{x,z} - \lambda v_{z,z} - \lambda' v_z + V' b_z + V b_{z,z} - u' b_x), \tag{3.17}
\end{aligned}$$

$$\frac{1}{\alpha} \frac{\partial b_z}{\partial t} = 0, \tag{3.18}$$

$$b_{z,z} = 0, \tag{3.19}$$

$$\begin{aligned}
& \rho \frac{\partial \tilde{\rho}}{\partial t} + p \frac{\partial \tilde{p}}{\partial t} + (\rho + p) \gamma^2 (V \frac{\partial v_x}{\partial t} + u \frac{\partial v_z}{\partial t}) + \alpha u \rho_{,z} + \alpha u p p_{,z} \\
& + \alpha (\rho + p) \{ \gamma^2 u V v_{x,z} + (1 + \gamma^2 u^2) v_{z,z} \} - \frac{1}{\gamma} (\tilde{\rho} - \tilde{p}) (\alpha u \gamma p)_{,z} \\
& + \alpha (\rho + p) \gamma^2 u \{ (1 + 2\gamma^2 V^2) V' + 2\gamma^2 u V u' \} v_x - \alpha (\rho + p) \\
& \times \{ (1 - 2\gamma^2 u^2) (1 + \gamma^2 u^2) \frac{u'}{u} \} - 2\gamma^4 u^2 V V' \} v_z = 0, \tag{3.20}
\end{aligned}$$

$$\begin{aligned}
& \left\{ (\rho + p) \gamma^2 (1 + \gamma^2 V^2) + \frac{B^2}{4\pi} \right\} \frac{1}{\alpha} \frac{\partial v_x}{\partial t} + \left\{ (\rho + p) \gamma^4 u V - \frac{\lambda B^2}{4\pi} \right\} \\
& \times \frac{1}{\alpha} \frac{\partial v_z}{\partial t} + \left\{ (\rho + p) \gamma^2 (1 + \gamma^2 V^2) + \frac{B^2}{4\pi} \right\} u v_{x,z} + \left\{ (\rho + p) \gamma^4 u V \right. \\
& \left. - \frac{\lambda B^2}{4\pi} \right\} u v_{z,z} - \frac{B^2}{4\pi} (1 + u^2) b_{x,z} - \frac{B^2}{4\pi \alpha} \{ \alpha' (1 + u^2) + \alpha u u' \} b_x \\
& + \gamma^2 u (\rho \tilde{\rho} + p \tilde{p}) \{ (1 + \gamma^2 V^2) V' + \gamma^2 u V u' \} + [(\rho + p) \gamma^4 u \{ (1 \\
& + 4\gamma^2 V^2) u u' + 4 V V' (1 + \gamma^2 V^2) \} + \frac{B^2 u \alpha'}{4\pi \alpha}] v_x + [(\rho + p) \gamma^2 \{ (1 \\
& + 2\gamma^2 u^2) (1 + 2\gamma^2 V^2) V' - \gamma^2 V^2 V' + 2\gamma^2 (1 + 2\gamma^2 u^2) u V u' \} \\
& - \frac{B^2 u}{4\pi \alpha} (\lambda \alpha)'] v_z = 0, \tag{3.21}
\end{aligned}$$

$$\begin{aligned}
& \left\{ (\rho + p) \gamma^2 (1 + \gamma^2 u^2) + \frac{\lambda^2 B^2}{4\pi} \right\} \frac{1}{\alpha} \frac{\partial v_z}{\partial t} + \left\{ (\rho + p) \gamma^4 u V - \frac{\lambda B^2}{4\pi} \right\} \\
& \times \frac{1}{\alpha} \frac{\partial v_x}{\partial t} + \left\{ (\rho + p) \gamma^2 (1 + \gamma^2 u^2) + \frac{\lambda^2 B^2}{4\pi} \right\} u v_{z,z} + \{ (\rho + p) \gamma^4 u \\
& \times V - \frac{\lambda B^2}{4\pi} \} u v_{x,z} + \frac{\lambda B^2}{4\pi} (1 + u^2) b_{x,z} + \frac{B^2}{4\pi \alpha} \{ \alpha' \lambda - (\alpha \lambda)' + u \lambda \\
& \times (u \alpha' + u' \alpha) \} b_x + (\rho \tilde{\rho} + p \tilde{p}) \gamma^2 \{ a_z + u u' (1 + \gamma^2 u^2) + \gamma^2 u^2 V V' \} \\
& + [(\rho + p) \gamma^4 \{ u^2 V' (1 + 4\gamma^2 V^2) + 2V (a_z + u u' (1 + 2\gamma^2 u^2)) \} - \lambda B^2 \\
& \times \frac{u \alpha'}{4\pi \alpha}] v_x + [(\rho + p) \gamma^2 \{ u' (1 + \gamma^2 u^2) (1 + 4\gamma^2 u^2) + 2u \gamma^2 (a_z + (1 \\
& + 2\gamma^2 u^2) V V' \} + \frac{\lambda B^2 u}{4\pi \alpha} (\alpha \lambda)'] v_z + (p' \tilde{p} + p \tilde{p}') = 0, \tag{3.22}
\end{aligned}$$

$$\begin{aligned}
& \frac{1}{\alpha}\gamma^2\rho\frac{\partial\tilde{\rho}}{\partial t} + \frac{1}{\alpha}\gamma^2p\frac{\partial\tilde{p}}{\partial t} + \gamma^2(\rho' + p')v_z + u\gamma^2(\rho\tilde{\rho}_{,z} + p\tilde{p}_{,z} + \rho'\tilde{\rho} \\
& + p'\tilde{p}) - \frac{1}{\alpha}p\frac{\partial\tilde{p}}{\partial t} + 2\gamma^2u(\rho\tilde{\rho} + p\tilde{p})a_z + \gamma^2u'(\rho\tilde{\rho} + p\tilde{p}) + 2(\rho \\
& + p)\gamma^4(uV' + 2uVa_z + u'V)v_x + 2(\rho + p)\gamma^2(2\gamma^2uu' + a_z\gamma^4 \\
& + 2\gamma^2u^2a_z)v_z + 2(\rho + p)\gamma^4uVv_{x,z} + (\rho + p)\gamma^2(1 + 2\gamma^2u^2) \\
& \times v_{z,z} - \frac{B^2}{4\pi\alpha}[(V^2 + u^2)\lambda b_x + (V^2 + u^2)b_z - \lambda V(\lambda V \\
& + u)\frac{\partial b_x}{\partial t} - u(\lambda V + u)\frac{\partial b_z}{\partial t}] - \frac{B^2}{4\pi\alpha}[(V - \lambda u)v_{x,t} + \lambda(u\lambda \\
& - V)v_{z,t}] + \frac{B}{4\pi}b_{x,z} = 0.
\end{aligned} \tag{3.23}$$

For Fourier analysis, we take the harmonic spacetime dependence of perturbation

$$\begin{aligned}
\tilde{\rho}(t, z) &= c_1 e^{-\iota(\omega t - kz)}, & \tilde{p}(t, z) &= c_2 e^{-\iota(\omega t - kz)}, \\
v_z(t, z) &= c_3 e^{-\iota(\omega t - kz)}, & v_x(t, z) &= c_4 e^{-\iota(\omega t - kz)}, \\
b_z(t, z) &= c_5 e^{-\iota(\omega t - kz)}, & b_x(t, z) &= c_6 e^{-\iota(\omega t - kz)}.
\end{aligned} \tag{3.24}$$

Here k is the z -component of the wave vector $(0, 0, k)$ and ω is the angular frequency. Using the wave vector, we obtain refractive index which helps to examine the behavior of plasma waves near the event horizon. Wave vector and dispersion relation can be defined as

- **Wave Vector:** A vector whose direction indicates the direction of phase propagation of a wave is called wave vector. Its magnitude is the wave number.
- **Dispersion Relation:** Dispersion relation gives the angular frequency ω as a function of wave vector k in the form $\frac{\omega}{k} = \lambda\nu$, where λ is the wavelength [27]. Dispersion is said to be anomalous if the refractive index is less than one and its change with respect to angular frequency is negative, otherwise normal.

The Fourier analyzed form of Eqs.(3.17)-(3.23), by using Eq.(3.24), are

$$c_4(\alpha' + \iota k \alpha) - c_3 \{(\alpha \lambda)' + \iota k \alpha \lambda\} - c_5(\alpha V)' + c_6\{(\alpha u)' + \iota \omega + \iota k u \alpha\} = 0, \quad (3.25)$$

$$c_5\left(\frac{-\iota \omega}{\alpha}\right) = 0, \quad (3.26)$$

$$c_5 \iota k = 0, \quad (3.27)$$

$$\begin{aligned} & c_1\{(-\iota \omega + \iota k \alpha u)\rho - p\gamma^2 \alpha u(VV' + uu') - \alpha' u p - \alpha u' p - \alpha u p'\} \\ & + c_2\{(-\iota \omega + \iota k \alpha u)p + \alpha' u p + \alpha u' p + \alpha u p' + p\gamma^2 \alpha u(VV' + uu')\} \\ & + c_3(\rho + p)[- \iota \omega \gamma^2 u + \iota k \alpha(1 + \gamma^2 u^2) - \alpha\{(1 - 2\gamma^2 u^2)(1 + \gamma^2 u^2) \\ & \times \frac{u'}{u} - 2\gamma^4 u^2 VV'\}] + c_4(\rho + p)[\gamma^2 V(-\iota \omega + \iota k \alpha u) + \alpha \gamma^2 u\{(1 \\ & + 2\gamma^2 V^2)V' + 2\gamma^2 u V u'\}] = 0, \end{aligned} \quad (3.28)$$

$$\begin{aligned} & c_1 \rho \gamma^2 u\{(1 + \gamma^2 V^2)V' + \gamma^2 u V u'\} + c_2 p \gamma^2 u\{(1 + \gamma^2 V^2)V' \\ & + \gamma^2 u V u'\} + c_3[-\{(\rho + p)\gamma^4 u V - \frac{\lambda B^2}{4\pi}\}\frac{\iota \omega}{\alpha} + \{(\rho + p)\gamma^4 u V \\ & - \frac{\lambda B^2}{4\pi}\}\iota k u + (\rho + p)\gamma^2\{(1 + 2\gamma^2 u^2)(1 + 2\gamma^2 V^2) - \gamma^2 V^2\}V' \\ & + 2\gamma^4(\rho + p)u V u'(1 + 2\gamma^2 u^2) - \frac{B^2 u}{4\pi \alpha}(\alpha \lambda)'] + c_4[-\{(\rho + p)\gamma^2 \\ & (1 + \gamma^2 V^2) + \frac{B^2}{4\pi}\}\frac{\iota \omega}{\alpha} + \{(\rho + p)\gamma^2(1 + \gamma^2 V^2) + \frac{B^2}{4\pi}\}\iota k u \\ & + (\rho + p)\gamma^4 u\{(1 + 4\gamma^2 V^2)uu' + 4VV'(1 + \gamma^2 V^2)\} + \frac{B^2 u \alpha'}{4\pi \alpha}] \\ & - c_6 \frac{B^2}{4\pi}\{(1 + u^2)\iota k + (1 + u^2)\frac{\alpha'}{\alpha} + uu'\} = 0, \end{aligned} \quad (3.29)$$

$$\begin{aligned} & c_1 \rho \gamma^2\{a_z + uu'(1 + \gamma^2 u^2) + \gamma^2 u^2 VV'\} + c_2[p\gamma^2\{a_z + uu' \\ & (1 + \gamma^2 u^2) + \gamma^2 u^2 VV'\} + p' + \iota k p] + c_3[-\{(\rho + p)\gamma^2(1 + \gamma^2 u^2) \\ & + \frac{\lambda^2 B^2}{4\pi}\}\frac{\iota \omega}{\alpha} + \{(\rho + p)\gamma^2(1 + \gamma^2 u^2) + \frac{\lambda^2 B^2}{4\pi}\}\iota k u + \{(\rho + p)\gamma^2 \\ & \times \{u'(1 + \gamma^2 u^2)(1 + 4\gamma^2 u^2) + 2u\gamma^2\{a_z + (1 + 2\gamma^2 u^2)\}VV'\} \\ & + \frac{\lambda B^2 u}{4\pi \alpha}(\alpha \lambda)'] + c_4[-\{(\rho + p)\gamma^4 u V - \frac{\lambda B^2}{4\pi}\}\frac{\iota \omega}{\alpha} + \{(\rho + p)\gamma^4 u V \end{aligned}$$

$$\begin{aligned}
& -\frac{\lambda B^2}{4\pi}\} \iota k u + \{(\rho + p)\gamma^4\{u^2 V'(1 + 4\gamma^2 V^2) + 2V\{(1 + 2\gamma^2 u^2)uu' \\
& + a_z - \frac{\lambda B^2 \alpha' u}{4\pi\alpha}\}\} + c_6[\frac{B^2}{4\pi\alpha}\{-(\alpha\lambda)' + \alpha'\lambda - u\lambda(u\alpha' + u'\alpha)\} \\
& + \frac{\lambda B^2}{4\pi}(1 + u^2)\iota k] = 0, \tag{3.30}
\end{aligned}$$

$$\begin{aligned}
& c_1\{(\frac{-\iota\omega}{\alpha}\gamma^2 + \iota k u \gamma^2 + 2u\gamma^2 a_z + \gamma^2 u')\rho + u\rho'\gamma^2\} + c_2\{(\frac{\iota\omega}{\alpha}(1 - \gamma^2) \\
& + \iota k u \gamma^2 + 2\gamma^2 u a_z + \gamma^2 u')p + u\gamma^2 p'\} + c_3\gamma^2\{(\rho' + p') + 2 \\
& \times (2\gamma^4 uu' + a_z + 2\gamma^2 u^2 a_z)(\rho + p) + (1 + 2\gamma^2 u^2)(\rho + p)\iota k + \frac{\lambda B^2}{4\pi\alpha} \\
& \times (\lambda u - V)\iota\omega\} + c_4[2(\rho + p)\gamma^2\{(uV' + 2uVa_z + u'V) + uV\iota k\} \\
& + \frac{B^2}{4\pi\alpha}(V - u\lambda)\iota\omega] + c_6[\frac{-B^2}{4\pi\alpha}\{(V^2 + u^2)\lambda + \lambda V(\lambda V + u)\iota\omega\} \\
& + \frac{B}{4\pi}\iota k] = 0. \tag{3.31}
\end{aligned}$$

These equations will be used to find dispersion relations.

4 Plasma Flow with Rotating Non-Magnetized Background

In the present section we shall study the rotating non-magnetized plasma, i.e., $\mathbf{B} = 0$. Equations (3.2) and (3.3) which are evolution equations of the magnetic field vanish. In the Fourier analyzed perturbed GRMHD equations (Eqs.(3.28)-(3.31)), we put $B = 0 = \lambda$ and $c_5 = 0 = c_6$ and obtain

$$\begin{aligned}
& c_1(\frac{-\iota\omega}{\alpha}\rho) + c_2(\frac{-\iota\omega}{\alpha}p) + c_3(\rho + p)[\frac{-\iota\omega}{\alpha}\gamma^2 u + (1 + \gamma^2 u^2)\iota k \\
& - (1 - 2\gamma^2 u^2)(1 + \gamma^2 u^2)\frac{u'}{u} + 2\gamma^4 u^2 VV'] + c_4(\rho + p)\gamma^2[(\frac{-\iota\omega}{\alpha} \\
& + \iota k u)V + u(1 + 2\gamma^2 V^2)V' + 2\gamma^2 u^2 V u'] = 0, \tag{4.1} \\
& c_1\rho\gamma^2 u\{(1 + \gamma^2 V^2)V' + \gamma^2 V u u'\} + c_2 p\gamma^2 u\{(1 + \gamma^2 V^2)V' \\
& + \gamma^2 V u u'\} + c_3(\rho + p)\gamma^2\{(\frac{-\iota\omega}{\alpha} + \iota k u)\gamma^2 V u + (1 + 2\gamma^2 u^2) \\
& \times (1 + 2\gamma^2 V^2)V' - \gamma^2 V^2 V' + 2\gamma^2(1 + 2\gamma^2 u^2)uV u'\} + c_4(\rho
\end{aligned}$$

$$+p)\gamma^2[(1+\gamma^2V^2)(\frac{-\iota\omega}{\alpha} + \iota ku) + \gamma^2u\{(1+4\gamma^2u^2)uu' + 4V$$

$$V'(1+\gamma^2V^2)\}] = 0, \quad (4.2)$$

$$c_1\rho\gamma^2\{a_z + (1+\gamma^2u^2)uu' + \gamma^2u^2VV'\} + c_2[p\gamma^2\{a_z + (1+\gamma^2u^2)$$

$$\times uu' + \gamma^2u^2VV'\} + (p' + \iota kp)] + c_3(\rho + p)\gamma^2[(1+\gamma^2u^2)(\frac{-\iota\omega}{\alpha}$$

$$+ \iota ku) + u'(1+\gamma^2u^2)(1+4\gamma^2u^2) + 2u\gamma^2\{a_z + (1+2\gamma^2u^2)$$

$$\times VV'\}] + c_4(\rho + p)\gamma^4[uV(\frac{-\iota\omega}{\alpha} + \iota ku) + u^2V'(1+4\gamma^2u^2)$$

$$+ 2V\{a_z + (1+2\gamma^2u^2)uu'\}] = 0, \quad (4.3)$$

$$c_1\{(\frac{-\iota\omega}{\alpha}\gamma^2 + \iota ku\gamma^2 + 2u\gamma^2a_z + \gamma^2u')\rho + u\rho'\gamma^2\} + c_2\{(\frac{\iota\omega}{\alpha}(1-\gamma^2)$$

$$+ \iota ku\gamma^2 + 2\gamma^2ua_z + \gamma^2u')p + u\gamma^2p'\} + c_3\gamma^2\{(\rho' + p') + 2(2\gamma^2uu'$$

$$+ a_z + 2\gamma^2u^2a_z)(\rho + p) + (1+2\gamma^2u^2)(\rho + p)\iota k\} + c_4(\rho + p)2\gamma^4$$

$$\{(uV' + 2uVa_z + u'V) + uV\iota k\} = 0. \quad (4.4)$$

4.1 Numerical Solutions

In order to determine numerical solutions, we use the the following assumptions

- Specific enthalpy: $\mu = 1$.
- Time lapse: $\alpha = \tanh(10z)/10$,
- Stationary fluid: $\alpha\gamma = 1$ with velocity components $V = u$ yields the following expression: $\alpha\gamma = 1 \Rightarrow \gamma = 1/\sqrt{1-u^2-V^2} = 1/\alpha$.
- Velocity components: $u = V$, x and z -components of velocity lead to $u = V = -\sqrt{\frac{1-\alpha^2}{2}}$.

With the assumption of stiff fluid, i.e., $\rho = p$, when we replace these values, the mass conservation law in three dimensions yields $\rho = p = -1/2u$. The GRMHD equations (Eqs.(3.2)-(3.6)) are satisfied by the above assumptions for the region $1.5 \leq z \leq 10, 0 \leq \omega \leq 10$. For plasma flow in rotating non-magnetized background, we acquire the constant values of $u = V = -0.703562$ which make the flow constants $l = 0.703562$ and $e = 1$. The determinant of the coefficients of constants of Eqs.(4.1)-(4.4) are solved which lead to a complex dispersion relation [28]. By comparison of real and

imaginary parts, two dispersion relations are found. The real part of the determinant gives an equation quartic in k

$$A_1(z)k^4 + A_2(z, \omega)k^3 + A_3(z, \omega)k^2 + A_4(z, \omega)k + A_5(z, \omega) = 0 \quad (4.5)$$

which gives two real values of k and two complex values conjugate. The equation obtained from the complex part is cubic in k

$$B_1(z)k^3 + B_2(z, \omega)k^2 + B_3(z, \omega)k + B_4(z, \omega) = 0 \quad (4.6)$$

which yields one real value of k and the remaining two are complex conjugate of each other. We can compute refractive index and its change with respect to angular frequency by using the real values obtained from Eq.(4.5) and (4.6). The results reached are shown in Figures 1, 2 and 3. The following table shows the results obtained from these Figures.

Table I. Direction and refractive index of waves.

| | Direction of Waves | Refractive Index (n) |
|----------|----------------------------------|---|
| 1 | Move towards the event horizon | $n < 1$ in the region $0 \leq z \leq 10, 8.5 \leq \omega \leq 10$ and $0 \leq z \leq 10, 9.5 \leq \omega \leq 10$ with the decrease in z |
| 2 | Move towards the event horizon | $n < 1$ in the region $2 \leq z \leq 10, 2 \leq \omega \leq 9$ and $2 \leq z \leq 10, 9.9 \leq \omega \leq 10$ with the decrease in z |
| 3 | Move away from the event horizon | $n < 1$ in the region $0 \leq z \leq 10, 2 \leq \omega \leq 10$ with the decrease in z |

In Figure 1, the change in refractive index with respect to angular frequency represents normal as well as anomalous dispersion of waves at random points. In Figure 2, dispersion is anomalous in the regions $2 \leq z \leq 10, 2.2 \leq \omega \leq 2.25$ and $0 \leq z \leq 9, 7.5 \leq \omega \leq 8$ while Figure 3 also shows normal and anomalous dispersion randomly. The group and phase velocities are found to be antiparallel in all figures.

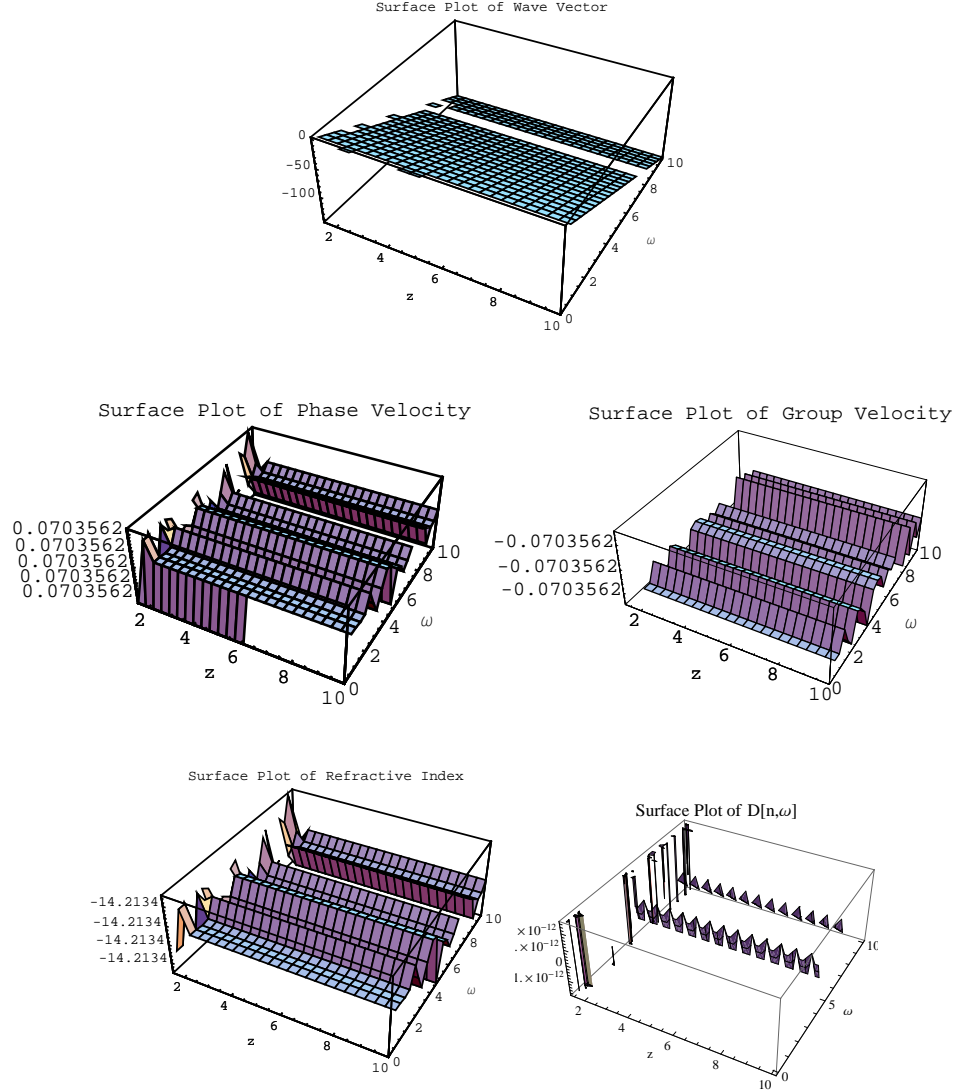


Figure 1: Waves move towards the event horizon. The dispersion is normal as well as anomalous at random points.

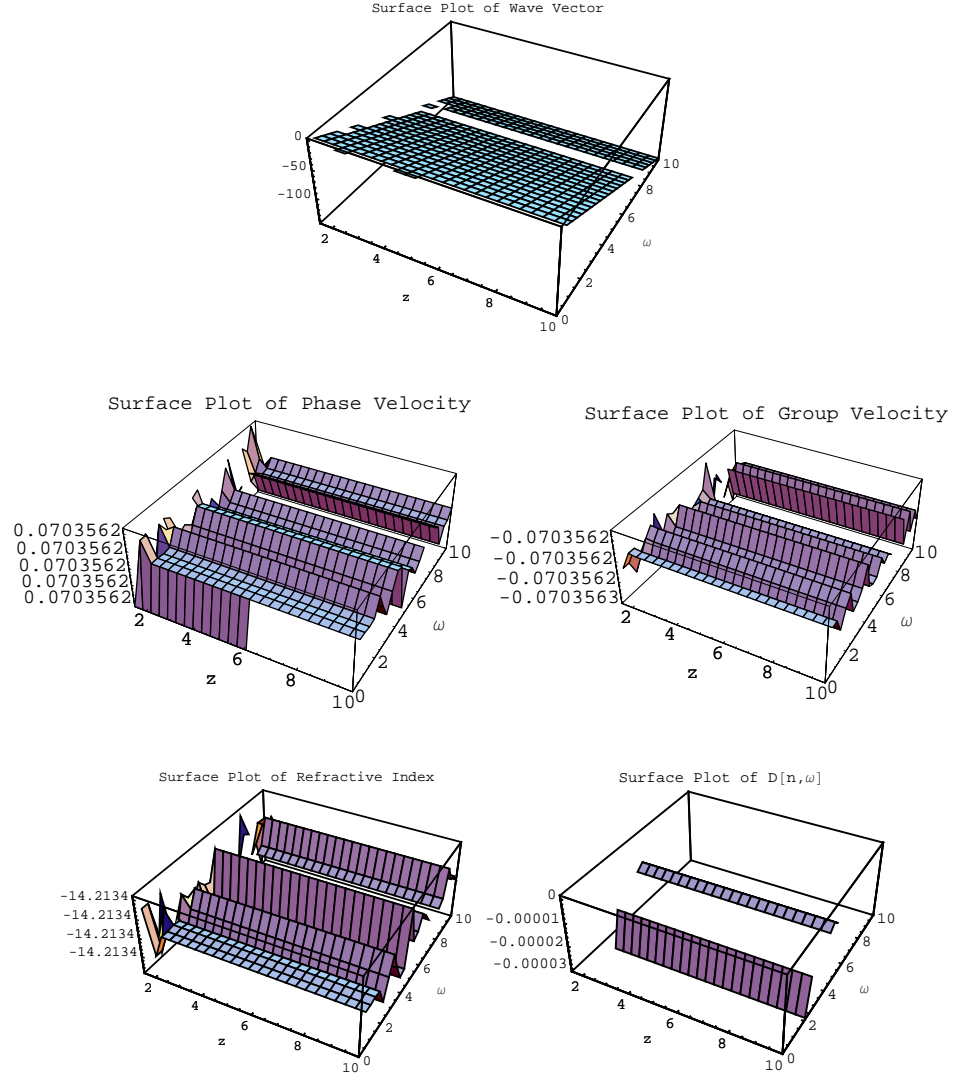


Figure 2: Waves are directed towards the event horizon. Region has anomalous dispersion.

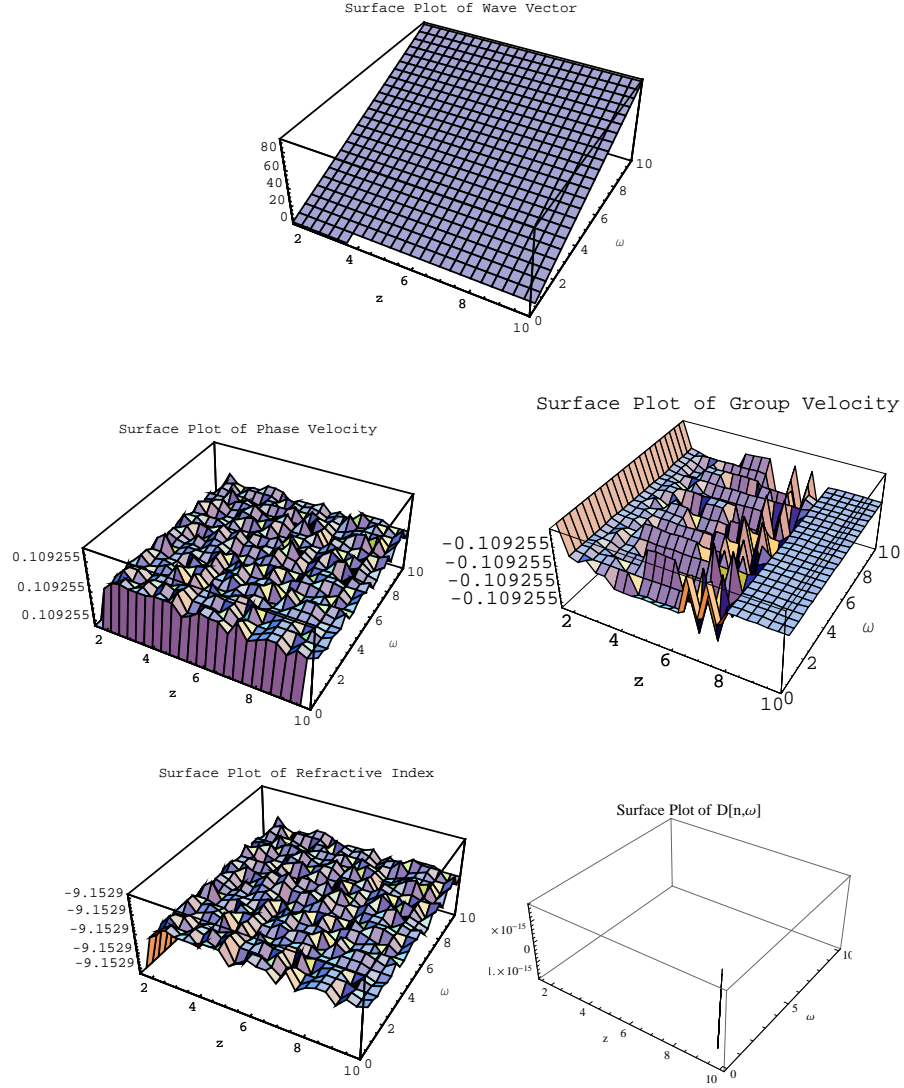


Figure 3: Waves are directed away from the event horizon. The region has normal and anomalous dispersion of waves.

5 Plasma Flow with Rotating Magnetized Background

This is the general case in which we assume that plasma is magnetized and rotating. We suppose xz -plane for the velocity and magnetic field of fluid. Equations (3.25)-(3.31) are the corresponding Fourier analyzed perturbed GRMHD equations.

5.1 Numerical Solutions

For magnetized background, we assume $V^F = 0$ with $u = V$ so that Eq.(3.8) gives $\lambda = 1$. Also, we consider $B = \sqrt{\frac{176}{7}}$ which leads to $\frac{B^2}{4\pi} = 2$. Further, the values of lapse function, velocity, pressure, density and specific enthalpy remain the same as given in section 4.1. The above assumed values satisfy the perfect GRMHD Eqs.(3.2)-(3.6) for the range $1.5 \leq z \leq 10$, $0 \leq \omega \leq 10$. The values of flow constants are $f = e = 1$, $l = -7.32001$ and $h = -1.76321$. Also, we have $u = V = -0.703562$. We obtain the determinant of the coefficients of constants by substituting these values in Eqs.(3.25) and (3.28)-(3.31) which gives two dispersion relations. Also, $c_5 = 0$ from Eqs.(3.26)-(3.27). The dispersion relation obtained from the real part has the following form

$$A_1(z)k^4 + A_2(z, \omega)k^3 + A_3(z, \omega)k^2 + A_4(z, \omega)k + A_5(z, \omega) = 0 \quad (5.1)$$

while the imaginary part gives the following dispersion relation

$$B_1(z)k^5 + B_2(z, \omega)k^4 + B_3(z, \omega)k^3 + B_4(z, \omega)k^2 + B_5(z, \omega)k + B_6(z, \omega) = 0. \quad (5.2)$$

We have used the software *Mathematica* to solve the above equations so that the roots can be displayed in terms of graphs. However, we could not find any graph of the roots. It seems that either all the roots are imaginary or there does not exist any wave in this region.

6 Summary

This paper investigates the isothermal plasma wave properties of the Schwarzschild magnetosphere in a Veselago medium. For this purpose, we have re-formulated

the Maxwell and $3 + 1$ GRMHD equations by considering both permittivity and permeability less than zero. The component and Fourier analysis of these equations are derived for rotating plasma in non-magnetized and magnetized backgrounds.

For the rotating non-magnetized plasma, our assumed values satisfy the $3 + 1$ GRMHD equations in the region $1.5 \leq z \leq 10$. In Figures **1** and **2**, waves move towards the event horizon. The dispersion is normal as well as anomalous at random points in the first figure while it is anomalous in the second figure. The third figure indicates that waves move towards the event horizon while it has normal and anomalous dispersion of waves randomly. All the figures show that phase and group velocities are antiparallel. Refractive index is less than one and it increases in a small region near the event horizon. From the previous literature [19] of the usual medium for isothermal plasma, we know that all waves move towards the event horizon. In the Veselago medium, Figure **3** indicates that waves move away from the event horizon. This is entirely a different result in this unusual medium. For the usual medium, the value of refractive index is always greater than one while it is less than one in all figures. This confirms the presence of Veselago medium. The rotating magnetized case does not provide any explicit graph which indicates that waves may not be found in the region $1.5 \leq z \leq 10$.

It would be interesting to extend this work by assuming hot plasma in this unusual medium. The Kerr spacetime can also be used to analyze the wave properties in this unusual medium.

Appendix A

In Veselago medium ($\epsilon < 0, \mu < 0$), Maxwell equations, the GRMHD equations for the general line element and the Schwarzschild planar analogue are given in this appendix. Maxwell equations in this unusual medium are

$$\nabla \cdot \mathbf{B} = 0, \tag{A1}$$

$$\nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0, \tag{A2}$$

$$\nabla \cdot \mathbf{E} = -\frac{\rho_e}{\epsilon}, \tag{A3}$$

$$\nabla \times \mathbf{B} = -\mu \mathbf{j} + \frac{\partial \mathbf{E}}{\partial t} = 0. \tag{A4}$$

The GRMHD equations in this medium will be

$$\frac{d\mathbf{B}}{d\tau} + \frac{1}{\alpha}(\mathbf{B} \cdot \nabla)\beta + \theta\mathbf{B} = -\frac{1}{\alpha}\nabla \times (\alpha\mathbf{V} \times \mathbf{B}), \quad (\text{A5})$$

$$\nabla \cdot \mathbf{B} = 0, \quad (\text{A6})$$

$$\frac{D\rho_0}{D\tau} + \rho_0\gamma^2\mathbf{V} \cdot \frac{D\mathbf{V}}{D\tau} + \frac{\rho_0}{\alpha} \left\{ \frac{g_{,t}}{2g} + \nabla \cdot (\alpha\mathbf{V} - \beta) \right\} = 0, \quad (\text{A7})$$

$$\begin{aligned} & \left\{ \left(\rho_0\mu\gamma^2 + \frac{\mathbf{B}^2}{4\pi} \right) \gamma_{ij} + \rho_0\mu\gamma^4 V_i V_j - \frac{1}{4\pi} B_i B_j \right\} \frac{DV^j}{D\tau} \\ & + \rho_0\gamma^2 V_i \frac{D\mu}{D\tau} - \left(\frac{\mathbf{B}^2}{4\pi} \gamma_{ij} - \frac{1}{4\pi} B_i B_j \right) V_{|k}^j V^k = -\rho_0\gamma^2 \mu \{ a_i \\ & - \frac{1}{\alpha} \beta_{j|i} V^j - (\mathcal{L}_t \gamma_{ij}) V^j \} - p_{|i} + \frac{1}{4\pi} (\mathbf{V} \times \mathbf{B})_i \nabla \cdot (\mathbf{V} \times \mathbf{B}) \\ & - \frac{1}{8\pi\alpha^2} (\alpha\mathbf{B})_{|i}^2 + \frac{1}{4\pi\alpha} (\alpha B_i)_{|j} B^j - \frac{1}{4\pi\alpha} (\mathbf{B} \times \{ \mathbf{V} \times [\nabla \\ & \times (\alpha\mathbf{V} \times \mathbf{B}) - (\mathbf{B} \cdot \nabla)\beta] + (\mathbf{V} \times \mathbf{B}) \cdot \nabla \beta \})_i, \end{aligned} \quad (\text{A8})$$

$$\begin{aligned} & \frac{D}{D\tau} (\mu\rho_0\gamma^2) - \frac{dp}{d\tau} + \Theta(\mu\rho_0\gamma^2 - p) + \frac{1}{2\alpha} (\mu\rho_0\gamma^2 V^i V^j \\ & + p\gamma_{ij}) \mathcal{L}_t \gamma_{ij} + 2\mu\rho_0\gamma^2 (\mathbf{V} \cdot \mathbf{a}) + \mu\rho_0\gamma^2 (\nabla \cdot \mathbf{V}) - \frac{1}{\alpha} \beta^{j,i} \\ & \times (\mu\rho_0\gamma^2 V_i V_j + p\gamma_{ij}) - \frac{1}{4\pi} (\mathbf{V} \times \mathbf{B}) \cdot (\mathbf{V} \times \frac{d\mathbf{B}}{d\tau}) - \frac{1}{4\pi} \\ & \times (\mathbf{V} \times \mathbf{B}) \cdot (\mathbf{B} \times \frac{d\mathbf{V}}{d\tau}) - \frac{1}{4\pi\alpha} (\mathbf{V} \times \mathbf{B}) \cdot \nabla \beta - \frac{1}{4\pi} \theta (\mathbf{V} \times \mathbf{B}) \\ & + \frac{1}{4\pi\alpha} (\nabla \times \alpha B) = 0. \end{aligned} \quad (\text{A9})$$

Since β , θ and $\mathcal{L}_t \gamma_{ij}$ vanish for the Schwarzschild planar analogue, the perfect GRMHD equations reduce to

$$\frac{\partial \mathbf{B}}{\partial t} = -\nabla \times (\alpha\mathbf{V} \times \mathbf{B}), \quad (\text{A10})$$

$$\nabla \cdot \mathbf{B} = 0, \quad (\text{A11})$$

$$\begin{aligned} & \frac{\partial \rho_0}{\partial t} + (\alpha\mathbf{V} \cdot \nabla) \rho_0 + \rho_0\gamma^2 \mathbf{V} \cdot \frac{\partial \mathbf{V}}{\partial t} + \rho_0\gamma^2 \mathbf{V} \cdot (\alpha\mathbf{V} \cdot \nabla) \mathbf{V} \\ & + \rho_0 \nabla \cdot (\alpha\mathbf{V}) = 0, \end{aligned} \quad (\text{A12})$$

$$\left\{ \left(\rho_0\mu\gamma^2 + \frac{\mathbf{B}^2}{4\pi} \right) \delta_{ij} + \rho_0\mu\gamma^4 V_i V_j - \frac{1}{4\pi} B_i B_j \right\} \left(\frac{1}{\alpha} \frac{\partial}{\partial t} + \mathbf{V} \cdot \nabla \right) V^j$$

$$\begin{aligned}
& -(\frac{\mathbf{B}^2}{4\pi}\delta_{ij} - \frac{1}{4\pi}B_iB_j)V^j{}_{,k}V^k + \rho_0\gamma^2V_i\{\frac{1}{\alpha}\frac{\partial\mu}{\partial t} + (\mathbf{V}\cdot\nabla)\mu\} \\
& = -\rho_0\mu\gamma^2a_i - p_{,i} + \frac{1}{4\pi}(\mathbf{V}\times\mathbf{B})_i\nabla\cdot(\mathbf{V}\times\mathbf{B}) - \frac{1}{8\pi\alpha^2}(\alpha\mathbf{B})^2{}_{,i} \\
& + \frac{1}{4\pi\alpha}(\alpha B_i)_{,j}B^j - \frac{1}{4\pi\alpha}[\mathbf{B}\times\{\mathbf{V}\times(\nabla\times(\alpha\mathbf{V}\times\mathbf{B}))\}]_i, \quad (\text{A13})
\end{aligned}$$

$$\begin{aligned}
& (\frac{1}{\alpha}\frac{\partial}{\partial t} + \mathbf{V}\cdot\nabla)(\mu\rho_0\gamma^2) - \frac{1}{\alpha}\frac{\partial p}{\partial t} + 2\mu\rho_0\gamma^2(\mathbf{V}\cdot\mathbf{a}) + \mu\rho_0\gamma^2(\nabla\cdot\mathbf{V}) \\
& - \frac{1}{4\pi}(\mathbf{V}\times\mathbf{B})\cdot(\mathbf{V}\times\frac{1}{\alpha}\frac{\partial\mathbf{B}}{\partial t}) - \frac{1}{4\pi}(\mathbf{V}\times\mathbf{B})\cdot(\mathbf{B}\times\frac{1}{\alpha}\frac{\partial\mathbf{V}}{\partial t}) \\
& + \frac{1}{4\pi\alpha}(\nabla\times\alpha\mathbf{B}) = 0. \quad (\text{A14})
\end{aligned}$$

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